A NOTE ON AN INVERSE SCATTERING PROBLEM FOR THE HELMHOLTZ EQUATION ON THE LINE

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Abstract. We consider the uniqueness in the inverse scattering problem for the equation

$$u'' + \frac{k^2}{c^2} = 0,$$

on $\mathbb{R}$, where c is a real measurable function with $c(x) \geq c_m > 0$, $|c(x) - 1| \leq C\langle x\rangle^{-1-\delta}$ and $c' \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

1. Introduction

We consider the inverse scattering problem for the equation

$$-u'' - \frac{k^2}{c^2}u = 0 \tag{1.1}$$

where c is a real measurable function satisfying

[H1] There exist $c_0 > 0$, $c_M > 0$ such $c_0 < c(x) \leq c_M$, a.e. $x \in \mathbb{R}$;

[H2] One has $|c(x) - 1| \leq C\langle x\rangle^{-1-\delta}$, a. e. $x \in \mathbb{R}$, with $\delta > 0$, and C is a positive constant independent of x.

Then for each $k \in \mathbb{R} \setminus \{0\}$ there exist unique solutions $u_1(x,k)$, $u_2(x,k)$ to the Helmholtz equation (1.1) such that

$$u_1(x,k) \sim \mathrm{e}^{\mathrm{i}kx} \quad \text{when} \quad x \to \infty,$$
$$u_2(x,k) \sim \mathrm{e}^{-\mathrm{i}kx} \quad \text{when} \quad x \to -\infty.$$

Then $u_1(\cdot,k)$ and $\overline{u_1(\cdot,k)}$ (respectively $u_2(\cdot,k)$ and $\overline{u_2(\cdot,k)}$) are linearly independent solutions of (1.1), therefore

$$u_1(x,k) \sim \frac{1}{T_2(k)}\mathrm{e}^{\mathrm{i}kx} + \frac{R_2(k)}{T_2(k)}\mathrm{e}^{-\mathrm{i}kx} \quad \text{when} \quad x \to -\infty,$$
$$u_2(x,k) \sim \frac{1}{T_1(k)}\mathrm{e}^{-\mathrm{i}kx} + \frac{R_1(k)}{T_1(k)}\mathrm{e}^{\mathrm{i}kx} \quad \text{when} \quad x \to \infty,$$

where $R_1(k)$, $R_2(k)$, $T_1(k)$ and $T_2(k)$ are complex constants detemined by c and k. (See also the next section.)

The matrix

$$S(k) = \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}$$

is the scattering matrix determined by c; $R_1(k)$ and $R_2(k)$ are the reflexion coefficients, whereas $T_1(k)$ and $T_2(k)$ are the transmission coefficients. $S(k)$ is an unitary matrix, and thus $|R_j(k)| \leq 1$, $j = 1, 2$.

We shall assume that $R_2(k) = R_2(k; c)$ is known. Our aim here is to prove the uniqueness part of the inverse scattering problem, that is, to prove the next theorem.

Theorem 1.1. *The mapping*

$$\mathcal{R}\colon \{c\colon \mathbb{R} \to \mathbb{R},\ \text{measurable} \ \mid c \text{ satisfies [H1] and [H2]},\ c' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\} \to L^\infty(\mathbb{R});$$
$$\mathcal{R}(c) = R_2(k; c)$$

is injective.

The methods used in the Schrödinger case ([F], [DT], [M], [AKM], and the references thererein) are not available in this case, since the behaviour in k, $|k|$ large, of the solutions $u_1(x,k)$, $u_2(x,k)$ is no longer easy to control, at least when c is not assumed to be smooth enough. To compensate, we shall use tools from the theory of analytic functions on a half-plane and an estimate ([SyWG] , [Br00]) for the reflexion coefficient associated to a function c which is constant on a half-axis and $c' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

The paper is organized as follows: In section 2 the construction of $u_1(x,k)$, $u_2(x,k)$ and of the scattering matrix is briefly presented, along with some of their properties. Section 3 deals with properties of the function $\frac{u_1'}{\mathrm{i}ku}$. The fact that $T_1(k)$ is uniquely determined by $R_2(k)$ is proved in section 4. Some further analysis of the behaviour in k of $u_1(x,k)$ is given in section 5. In all these sections only [H1] and [H2] are assumed to be satisfied. Finally, Theorem 1.1 is proved in section 6, under the additional assumption that $c' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

2. Jost solutions and the scattering matrix

2.1. **Jost solutions.** The scattering matrix is constructed by using special solutions of the Helmholtz equation. For $k \in \overline{\mathbb{C}^+}$ we consider the Jost solutions $(u_j)_{j=1,2}$ to the equation

$$u'' + \frac{k^2}{c^2}u = 0. \tag{2.1}$$

They satisfy $u_1(x,k) \sim \mathrm{e}^{\mathrm{i}kx}$ when $x \to \infty$, while $u_2(x,k) \sim \mathrm{e}^{-\mathrm{i}kx}$ when $x \to -\infty$.

The proof of the next theorem is similar to the proof of the analogous result in the Schrödinger case (see [DT]). We use the notation

$$q = 1 - \frac{1}{c^2}. \tag{2.2}$$

From [H1] and [H2] we see that

$$|q(x)| \leq C\langle x\rangle^{-1-\delta}, \quad a.e.\ x \in \mathbb{R}.$$

We also need the following functions

$$\gamma(x) = \int\limits_x^\infty |q(y)|\,\mathrm{d}y; \quad \eta(x) = \int\limits_\infty^x |q(y)|\,\mathrm{d}y.$$

and set

$$\gamma_0 = \sup_{x\in\mathbb{R}} |q(x)|. \tag{2.3}$$

Theorem 2.1. *For every $k \in \overline{\mathbb{C}^+}$, there exist unique the solutions $u_1(x,k)$, $u_2(x,k)$ to the equation* (2.1) *such that:*

$$\mathrm{e}^{-\mathrm{i}kx}u_1(x,k) \to 1, \quad \mathrm{e}^{\mathrm{i}kx}u_2(x,k) \to 1, \tag{2.4}$$

when $x \to \infty$ and $x \to -\infty$, respectively.

The following estimates hold:

$$|u_1(x,k) - \mathrm{e}^{\mathrm{i}kx}| \le |k|\gamma(x)\mathrm{e}^{|k|\gamma(x) - x\operatorname{Im} k}, \tag{2.5}$$

$$|u_2(x,k) - \mathrm{e}^{-\mathrm{i}kx}| \le |k|\eta(x)\mathrm{e}^{|k|\eta(x) + x\operatorname{Im} k}, \tag{2.6}$$

$$|u_1'(x,k) - \mathrm{i}ku_1(x,k)| \le |k|^2\gamma(x)\left(1 + |k|\|q\|_{L^1}\mathrm{e}^{|k|\|q\|_{L^1}}\right)\mathrm{e}^{-x\operatorname{Im} k}; \tag{2.7}$$

$$|u_2'(x,k) + \mathrm{i}ku_2(x,k)| \le |k|^2\eta(x)\left(1 + |k|\|q\|_{L^1}\mathrm{e}^{|k|\|q\|_{L^1}}\right)\mathrm{e}^{x\operatorname{Im} k}; \tag{2.8}$$

$$\overline{u_1(x,k)} = u_1(x,-\overline{k}), \qquad \overline{u_2(x,k)} = u_2(x,-\overline{k}). \tag{2.9}$$

When $x \in \mathbb{R}$ is fixed, the functions

$$\mathbb{C}^+ \ni k \mapsto u_j(x,k) \in \mathbb{C}, \qquad \mathbb{C}^+ \ni k \mapsto u_j'(x,k) \in \mathbb{C},$$

$j = 1, 2$, are holomorphic and extend continuously at $\operatorname{Im} k = 0$.

Remark. If we set

$$m_1(x,k) = \mathrm{e}^{-\mathrm{i}xk}u_1(x,k), \quad m_1(x,k) = \mathrm{e}^{-\mathrm{i}xk}u_1(x,k)$$

then $m_1(x,k)$ and $m_2(x,k)$ solve the equations

$$m_1''(x,k) + 2\mathrm{i}km_1'(x,k) = k^2q(x)m_1(x,k), \tag{2.10}$$

$$m_2''(x,k) - 2\mathrm{i}km_2'(x,k) = k^2q(x)m_2(x,k), \tag{2.11}$$

respectively. The previous theorem gives also that

$$|m_1(x,k) - 1| \le |k|\gamma(x)\mathrm{e}^{|k|\gamma(x)},$$
$$|m_2(x,k) - 1| \le |k|\eta(x)\mathrm{e}^{|k|\eta(x)},$$
$$|m_1'(x,k)| \le |k|^2\gamma(x)\left(1 + |k|\|q\|_{L^1}\mathrm{e}^{|k|\|q\|_{L^1}}\right),$$
$$|m_2'(x,k)| \le |k|^2\eta(x)\left(1 + |k|\|q\|_{L^1}\mathrm{e}^{|k|\|q\|_{L^1}}\right)$$

when $k \in \overline{\mathbb{C}^+}$ and $x \in \mathbb{R}$.

2.2. **The scattering matrix.** We denote by

$$[f, g]_x = f'(x)g(x) - f(x)g'(x)$$

the wronskian of f and g. If this quantity is constant on $\mathbb{R}$, it will be denoted simply by $[f, g]$.

Let $u_1(\cdot, k)$, $u_2(\cdot, k)$ be the solutions constructed in the previous subsection.

Consider $k \in \mathbb{R} \setminus \{0\}$. It follows from relations (2.5) and (2.7) Liouville's theorem that

$$\begin{aligned} [u_1(\cdot, k), u_1(\cdot, -k)] &= \lim_{x\to\infty} (u_1'(x, k)u_1(x, -k) - u_1(x, k)u_1'(x, -k)) \\ &= 2\mathrm{i}k \neq 0. \end{aligned} \tag{2.12}$$

Consequently, if $k \in \mathbb{R} \setminus \{0\}$ and a is positive, then

$$|au_1' \pm \mathrm{i}ku_1|^2 = a^2|u_1'|^2 + k^2|u_1|^2 \pm 2k^2a. \tag{2.13}$$

We similarly get (from (2.6) and (2.8))

$$[u_2(\cdot, k), u_2(\cdot, -k)] = -2\mathrm{i}k \neq 0, \tag{2.14}$$

$$|au_2' \pm \mathrm{i}ku_2|^2 = a^2|u_1'|^2 + k^2|u_2|^2 \mp 2k^2a. \tag{2.15}$$

We deduce from (2.9), that u_1, $\overline{u}_1$ (u_2, $\overline{u}_2$) are linearly independent solutions of (2.1). Using again (2.12) we see that there are some constants $T_1(k)$, $T_2(k)$, $R_1(k)$, $R_2(k)$ such that $1/T_j(k) \neq 0$, $j = 1, 2$ and

$$u_2(x, k) = \frac{R_1(k)}{T_1(k)} u_1(x, k) + \frac{1}{T_1(k)} u_1(x, -k), \tag{2.16}$$

$$u_1(x, k) = \frac{R_2(k)}{T_2(k)} u_2(x, k) + \frac{1}{T_2(k)} u_2(x, -k), \tag{2.17}$$

when $k \neq 0$ is real. $R_1(k)$, $R_2(k)$, $T_1(k)$, $T_2(k)$ are the scattering coefficients of c at energy k^2. The first two quantities are the reflexion coefficients, while the other two are the transmission coefficients, and the matrix

$$S(k) = \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}$$

is the scattering matrix.

The properties of u_1 and u_2 give more on the scattering coefficients. We see from (2.12), (2.14), (2.16) and (2.17) that

$$[u_1(x, k), u_2(x, k)] = \frac{2\mathrm{i}k}{T_1(k)} = \frac{2\mathrm{i}k}{T_2(k)}, \tag{2.18}$$

$$[u_2(x, k), u_1(x, -k)] = 2\mathrm{i}k \frac{R_1(k)}{T_1(k)}, \tag{2.19}$$

$$[u_2(x, -k), u_1(x, k)] = 2\mathrm{i}k \frac{R_2(k)}{T_2(k)}. \tag{2.20}$$

It follows from (2.18) and (2.15) that

$$T_1(k) = T_2(k) \overset{\text{not.}}{=\!=} T(k) \quad \text{and} \quad \overline{T(k)} = T(-k), \quad \text{when } k \in \mathbb{R}, \tag{2.21}$$

and then from (2.19), (2.20), (2.21) and (2.9) it follows that

$$R_1(k)T(-k) + R_2(-k)T(k) = 0, \tag{2.22}$$

$$\overline{R_1(k)} = R_1(-k), \quad \overline{R_2(k)} = R_2(-k). \tag{2.23}$$

Remark. The function $2ik/T(k)$ can be analytically extended to $\mathbb{C}^+$, taking (2.18) as a definition. It extends continuously to $\mathbb{R}\backslash\{0\}$, and it can not vanish in any point of $\overline{C^+}\backslash\{0\}$, since there are no eigenvalues for the operator $-c(cu)''$ considered as a self-adjoint operator in $L^2(\mathbb{R})$ with domain $\{u \in L^2 \mid (cu)'' \in L^2\}$. We thus see that $T(k)$ can be extended as a holomorphic function to $\mathbb{C}^+$, continuous on $\mathbb{R}^+ \setminus \{0\}$.

In (2.16) we replace $u_1(x, k)$ by its expression in (2.17) and use (2.22), (2.23). We get that

$$|T(k)|^2 + |R_2(k)|^2 = 1. \tag{2.24}$$

It follows similarly that

$$|T(k)|^2 + |R_1(k)|^2 = 1. \tag{2.25}$$

As a consequence, the matrix $S(k)$ is unitary and

$$|T(k)| \le 1, \quad |R_1(k)| \le 1, \quad |R_2(k)| \le 1, \quad k \in \mathbb{R} \setminus \{0\}. \tag{2.26}$$

We need an estimate for $R_2(k)$ when k is in a neighbourhood of 0. We can easily obtain (see Ch. 2,§3 [DT]):

$$\frac{R_2(k)}{T(k)} = \frac{k}{2\mathrm{i}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{2\mathrm{i}kt}\, q(t)\, m_1(t, k)\, \mathrm{d}t, \tag{2.27}$$

$$\frac{1}{T(k)} = 1 - \frac{k}{2\mathrm{i}} \int\limits_{-\infty}^{+\infty} q(t)\, m_1(t, k)\, \mathrm{d}t. \tag{2.28}$$

when $k \in \mathbb{R} \setminus \{0\}$.

Lemma 2.2. *(i) The function $\mathbb{R} \setminus \{0\} \ni k \mapsto T(k) \in \mathbb{C}$ extends to a continuous function on $\overline{C^+}$ with $T(0) = 1$.*

(ii) The function $\mathbb{R} \setminus \{0\} \ni k \mapsto R_2(k) \in \mathbb{C}$ extends to a continuous function on $\mathbb{R}$, with $R_2(0) = 0$. Moreover

$$|R_2(k)| \le C|k|, \quad \textit{when} \quad k \in \mathbb{R}, \tag{2.29}$$

$$\lim_{k \to 0} \frac{R_2(k)}{k} = \frac{1}{2\mathrm{i}} \int\limits_{-\infty}^{+\infty} q(t)\, \mathrm{d}t, \tag{2.30}$$

with C a constant independent on k.

Proof. The first part of the proof follows from the remark above, relation (2.28), and Theorem 2.1.

It follows form (2.27) that $k \to R_2(k)/T(k)$ extends continuously $\mathbb{R}$, and since $k \to T(k)$ extends continuously on $\mathbb{R}$ we get that $k \to R_2(k)$ has the same property. The remaining assertions of the statement are direct consequences of (2.27) and (2.28), taking into account that $|T(k)| \le 1$ and (2.5) holds. □

Lemma 2.3. *The equality*

$$2\mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k)) = |T(k)m_1(x,k)|^2 + |T(k)m_2(x,k)|^2. \tag{2.31}$$

holds for every $x \in \mathbb{R}$ and $k \in \mathbb{R}$.

Proof. We write $u_1(x,k) = \mathrm{e}^{\mathrm{i}kx}m_1(x,k)$ and $u_2(x,k) = \mathrm{e}^{-\mathrm{i}kx}m_2(x,k)$. Then the equality (2.16) becomes

$$T(k)m_2(x,k) = R_1(k)\mathrm{e}^{2\mathrm{i}kx}m_1(x,k) + \overline{m_1(x,k)}, \tag{2.32}$$

when $x \in \mathbb{R}$ and $k \in \mathbb{R}$. We have used that $\overline{m_1(x,k)} = m_1(x,-k)$ for k real. Thus

$$|T(k)m_2(x,k)|^2 = |m_1(x,k)|^2 + |R_1(k)m_1(x,k)|^2 + 2\mathrm{Re}\,(R_1(k)\mathrm{e}^{-2\mathrm{i}kx}m_1^2(x,k). \tag{2.33}$$

On the other hand, multiplying (2.32) by $m_1(x,k)$ and then taking real parts it follows that

$$2\mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k)) = 2|m_1(x,k)|^2 + 2\mathrm{Re}\,(R_1(k)\mathrm{e}^{-2\mathrm{i}kx}m_1^2(x,k)). \tag{2.34}$$

Subtracting this equality from (2.33) we get

$$|T(k)m_2(x,k)|^2 - 2\mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k)) = -|m_1(x,k)|^2 + |R_1(k)m_1(x,k)|^2.$$

Relation (2.31) is now a consequence of (2.25) □

3. The functions w and r

We need the functions defined by

$$w(x,k) = \begin{cases} \frac{u_1'(x,k)}{\mathrm{i}ku_1(x,k)}, & \text{when} \quad k \in \overline{\mathbb{C}}^+ \setminus \{0\} \\ 1, & \text{when} \quad k = 0 \end{cases}, \quad x \in \mathbb{R}, \tag{3.1}$$

and

$$r(x,k) = \frac{1 - w(x,k)}{1 + w(x,k)}, \quad x \in \mathbb{R},\ k \in \overline{\mathbb{C}^+}, \tag{3.2}$$

We first need to show that the definitions (3.2) and (3.1) make sense.

Lemma 3.1. *Let $k \in \overline{\mathbb{C}^+}$ be fixed. Then $u_1(x,k) \neq 0$ when $x \in \mathbb{R}$.*

Proof. We prove first that $u_1(x,k) \neq 0$. Assume $u_1(x_0,k) = 0$ for an $x_0 \in \mathbb{R}$ (x_0 may depend on k). When $k \in \mathbb{R}\setminus\{0\}$, this can not happen owning to (2.12) and (2.9). If $\operatorname{Im} k \neq 0$, then $\frac{1}{c}u_1(\cdot,k)$ is an eigenfunction of H_D, corresponding to the eigenvalue k^2, where $H_D f = -c(c\,f)''$ when $f \in D(H_D) = \{g \in L^2(x_0,\infty) \mid -c(c\,g)'' \in L^2(x_0,\infty),\ (cg)(x_0) = 0\}$. Thus, $k^2 \in \sigma(H_D)$, which contradicts the fact that $H_D \geq 0$. In view of (2.5), $u_1(x,0) = 1$, for any x real, and this concludes the proof of the lemma. $\square$

The previous result allows us to define

$$w(x,k) = \frac{u_1'(x,k)}{\mathrm{i}k u_1(x,k)}, \qquad x \in \mathbb{R},\ k \in \overline{\mathbb{C}^+}\setminus\{0\}. \tag{3.3}$$

In addition, we have (by (2.7), (2.5)),

$$\lim_{k\to 0} w(x,k) = \lim_{k\to 0} \frac{u_1'(x,k) - \mathrm{i}k u_1(x,k)}{\mathrm{i}k u_1(x,k)} + 1 = 1$$

Hence the function $w(\cdot,k)$ defined in (3.1) belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathbb{R})$, and if $x \in \mathbb{R}$ is fixed, the function $\mathbb{C}^+ \mapsto w(x,k) \in \mathbb{C}$ is analytic and extends continuously to $\operatorname{Im} k = 0$. (Thm. 2.1 and lemma 3.1). One can easily see that w satisfies

$$w'(x,k) = \frac{\mathrm{i}k}{c^2(x)} - \mathrm{i}k w^2(x,k). \tag{3.4}$$

It follows that $w'(\cdot,k)$ is locally bounded and w is absolutely continuous..

In a similar way we may define

$$w_-(x,k) = -\frac{u_2'(x,k)}{\mathrm{i}k u_2(x,k)}, \qquad \text{when} \quad x \in \mathbb{R},\ k \in \overline{\mathbb{C}^+}\setminus\{0\}, \tag{3.5}$$

and it has properties similar to those of w.

The next lemma contains a simple computation which turns out to be important in the sequel.

Lemma 3.2. (a) *When $k \in \mathbb{R}$, we have*

$$\operatorname{Re} w(x,k) = \frac{1}{|u_1(x,k)|^2}, \quad x \in \mathbb{R}. \tag{3.6}$$

(b) *When $\operatorname{Im} k > 0$, we have*

$$\operatorname{Re} w(x,k) = \frac{\operatorname{Im} k}{|k^2|\,|u_1(x,k)|^2} \int_x^\infty \left(\frac{|k|^2 |u_1(y,k)|^2}{c^2(y)} + |u_1'(y,k)|^2 \right) \mathrm{d}y \tag{3.7}$$

for $x \in \mathbb{R}$.

Thus $\operatorname{Re} w(x,k) > 0$ when $x \in \mathbb{R}$ and $k \in \mathbb{C}$, $\operatorname{Im} k \geq 0$.

Proof. (a) If $k \in \mathbb{R}\setminus\{0\}$, the equality (3.6) is a consequence of (2.9) and (2.12). The equality is obvious when $k = 0$, both sides being equal to 1.

(b) Assume now that $\operatorname{Im} k > 0$. We have

$$\operatorname{Re} w = \frac{1}{2\mathrm{i}|k|^2\,|u_1|^2}(u_1'\,\overline{u_1}\,\overline{k} - \overline{u_1'}\,u_1\,k). \tag{3.8}$$

On the other hand, since u_1 solves (2.1) and u_1 and u_1' decay exponentially when $x \to \infty$, we may write

$$\begin{aligned}(\overline{u_1}u_1')(x,k) &= -\int_x^\infty \overline{u_1}(y,k)\,u_1''(y,k)\,\mathrm{d}y - \int_x^\infty |u_1'(y,k)|^2\,\mathrm{d}y \\ &= \int_x^\infty \left(\frac{k^2|u_1|^2(y,k)}{c^2(y)} - |u_1'(y,k)|^2\right)\,\mathrm{d}y.\end{aligned}$$

Using this formula in (3.8) we get (3.7). □

Remark. One may prove similarly that

$$\operatorname{Re} w_-(x,k) > 0 \quad \text{on } \overline{\mathbb{C}}^+.$$

Corollary 3.3. *Consider $x \in \mathbb{R}$ be arbirary fixed. Then the function*

$$\mathbb{R} \ni k \to \frac{1}{1+k^2}\operatorname{Re}\frac{1}{w(x,k)+w_-(x,k)} \in \mathbb{R}$$

is absolutely summable on $\mathbb{R}$.

Proof. This is a consequence of the fact that

$$\operatorname{Re}\frac{1}{w(x,k)+w_-(x,k)} > 0, \quad k \in \mathbb{C}^+,$$

$k \to (w(x,k)+w_-(x,k))^{-1}$ is holomorphic on C^+ and extends continuously to $\operatorname{Im} k = 0$. (See [AD].) □

Proposition 3.4. *When $x \in \mathbb{R}$ and $\kappa > 0$ one has*

$$\frac{2}{2+c_M^2\gamma_0} \leq w(x,\mathrm{i}\kappa) \leq 1 + \frac{1}{2}\gamma_0. \tag{3.9}$$

Remark. We notice that $w(x,\mathrm{i}\kappa)$ is real when when $x \in \mathbb{R}$ and $\kappa > 0$ since $\overline{u_1(x,\mathrm{i}\kappa)} = u_1(x,\mathrm{i}\kappa)$. From lemma 3.2 we also see that $w(x,\mathrm{i}\kappa)$ is positive.

Proof. We set $v = v(\cdot,\mathrm{i}\kappa) = w(\cdot,\mathrm{i}\kappa) - 1$. We get then from (3.4) and (2.5) that

$$v(x,\mathrm{i}\kappa) \to 0 \quad \text{when} \quad x \to +\infty,$$

$$v' = \kappa q + \kappa v^2 + 2\kappa v \geq \kappa q + 2\kappa v.$$

It follows that

$$v(x,\mathrm{i}\kappa) \leq -\int_x^\infty \kappa q(y)\,\mathrm{e}^{-2\kappa(y-x)}\,\mathrm{d}y \leq \frac{1}{2}\sup_{[x,\infty)}|q| \leq \frac{1}{2}\gamma_0, \tag{3.10}$$

hence

$$w(x,\mathrm{i}\kappa) \leq 1 + \frac{1}{2}\gamma_0. \tag{3.11}$$

On the other hand we notice that $\tilde{w} = \tilde{w}(x, \mathrm{i}\kappa) = \frac{1}{w(x,\mathrm{i}\kappa)}$ $(w(x, \mathrm{i}\kappa) > 0)$ satisfies

$$\tilde{w}' = \frac{\kappa}{c^2}\tilde{w}^2 - \kappa,$$

and $\tilde{w}(x, \mathrm{i}\kappa) \to 1$ when $x \to \infty$. We set $\tilde{v} = \tilde{w} - 1$ and get

$$\tilde{v}' = \frac{\kappa}{c^2}\tilde{v}^2 + \frac{2\kappa}{c^2}\tilde{v} - \kappa q \geq \frac{2\kappa}{c^2}\tilde{v} - \kappa q.$$

Thus

$$\begin{aligned}\tilde{v}(x, \mathrm{i}\kappa) &\leq \int_x^\infty \kappa q(y)\, \mathrm{e}^{-2\kappa \int_x^y \frac{1}{c^2(s)}\,\mathrm{d}s}\, \mathrm{d}y \\ &\leq \int_x^\infty \kappa |q(y)|\, \mathrm{e}^{-2\kappa(y-x)/c_M^2}\, \mathrm{d}y \\ &\leq \frac{1}{2}c_M^2\gamma_0,\end{aligned}$$

and we obtain

$$\frac{1}{w} \leq 1 + \frac{1}{2}c_M^2\gamma_0. \tag{3.12}$$

The inequality (3.9) follows directly from (3.11) and (3.12). □

Lemma 3.5. *Let c_1 and c_2 be two measurable real functions obeying [H1] and [H2], and let q_1, q_2 and w_1, w_2 be the functions defined by* (2.2), (3.3), *corresponding to c_1 and c_2, respectively. Then*

$$\|(w_1 - w_2)(\cdot, \mathrm{i}\kappa)\|_{L^1(\mathbb{R})} \leq \frac{1}{\alpha}\|q_1 - q_2\|_{L^1(\mathbb{R})}, \quad \textit{when} \quad \kappa \geq 0,$$

where

$$\alpha = \frac{2}{2 + c_{M,1}^2\gamma_{0,1}} + \frac{2}{2 + c_{M,2}^2\gamma_{0,2}},$$

$c_{M,j}$ are the constant in [H1], corresponding to c_j, $j = 1, 2$, while $\gamma_{0,j} = \sup |q_j|$, $j = 1, 2$.

Proof. We see that $w_1(\cdot, \mathrm{i}\kappa) - w_2(\cdot, \mathrm{i}\kappa)$ satisfies

$$(w_1(\cdot, \mathrm{i}\kappa) - w_2(\cdot, \mathrm{i}\kappa))' = \kappa(q_2 - q_1) + \kappa(w_1(\cdot, \mathrm{i}\kappa) - w_2(\cdot, \mathrm{i}\kappa))(w_1(\cdot, \mathrm{i}\kappa) + w_2(\cdot, \mathrm{i}\kappa)),$$

$$w_1(x, \mathrm{i}\kappa) - w_2(x, \mathrm{i}\kappa) \to 0 \quad \text{when} \quad x \to \infty.$$

Hence we get

$$(w_1 - w_2)(x, \mathrm{i}\kappa) = \kappa \int_x^\infty (q_1 - q_2)(y)\mathrm{e}^{-\kappa \int_x^y (w_1 + w_2)(s, \mathrm{i}\kappa)\,\mathrm{d}s}\, \mathrm{d}y.$$

Proposition 3.4 ensures that

$$(w_1 + w_2)(s, \mathrm{i}\kappa) \geq \alpha$$

hence

$$(w_1 - w_2)(x, \mathrm{i}\kappa) \leq \kappa \int_x^\infty |(q_1 - q_2)(y)|\mathrm{e}^{-\alpha\kappa(y-x)}\, \mathrm{d}y.$$

We repeat this procedure for $(w_2 - w_1)(\cdot, \mathrm{i}\kappa)$ and get

$$|(w_1 - w_2)(x, \mathrm{i}\kappa)| \le \kappa \int_x^\infty |(q_1 - q_2)(y)| \mathrm{e}^{-\alpha\kappa(y-x)} \,\mathrm{d}y.$$

We integrate this inequality with respect to x and obtain

$$\begin{aligned}\int_{\mathbb{R}} |(w_1 - w_2)(x, \mathrm{i}\kappa)| \,\mathrm{d}x &\le \kappa \int_{\mathbb{R}} \int_0^\infty |(q_1 - q_2)(x+t)| \mathrm{e}^{-\alpha\kappa t} \,\mathrm{d}t \,\mathrm{d}x \\ &= \frac{1}{\alpha} \|q_1 - q_2\|_{L^1(\mathbb{R})},\end{aligned}$$

which completes the proof. □

Lemma 3.1 ensures that $\mathrm{Re}\,(1 + w(x,k)) > 1$, therefore $r(x,k)$ may be defined by (3.2), and the function $r(\cdot, k)$ is continuous when $k \in \overline{\mathbb{C}^+}$ is fixed.

The basic properties of r are contained in the next proposition.

Proposition 3.6. *The function $r(\cdot, k)$ defined by (3.2) satifies:*

(i) $|r(\cdot, k)| < 1$ *when* $x \in \mathbb{R}$, $k \in \overline{\mathbb{C}^+}$.

(ii) *When $k \in \overline{\mathbb{C}}^+$ is fixed, $r(\cdot, k)$ belongs to te Sobolev space Sobolev $W^{1,\infty}(\mathbb{R})$ and*

$$r' = -2\mathrm{i}k\, r + \frac{\mathrm{i}kq}{2}(1 + r)^2 \tag{3.13}$$

while

$$\lim_{x\to\infty} r(x,k) = 0 \quad \textit{when} \quad k \in \overline{\mathbb{C}}^+, \tag{3.14}$$

$$\lim_{x\to-\infty} \mathrm{e}^{2\mathrm{i}kx} r(x,k) = R_2(k) \quad \textit{when} \quad k \in \mathbb{R}. \tag{3.15}$$

(iii) *If $x \in \mathbb{R}$ is arbitrarily fixed, the function $\mathbb{C}^+ \ni k \mapsto r(x,k) \in \mathbb{C}$ is holomorphic and extends continuously at* $\mathrm{Im}\, k = 0$.

Proof. (i) The inequality is straightforward since

$$|r|^2 = \frac{(1 - \mathrm{Re}\, w)^2 + (\mathrm{Im}\, w)^2}{(1 + \mathrm{Re}\, w)^2 + (\mathrm{Im}\, w)^2},$$

and $\mathrm{Re}\, w > 0$ (Lemma 3.2).

(ii) We have

$$r' = -\frac{2}{(1+w)^2} w', \tag{3.16}$$

w' is locally bounded, $|1 + w| > 1$, hence $r'(\cdot, k)$ is locally bounded . The relation (3.13) follows from (3.16) and form the equation for w'. Using (3.13) and the inequality (i), we see that $r(\cdot, k) \in W^{1,\infty}(\mathbb{R})$.

We now prove (3.14). We write

$$r(x,k) = \frac{\mathrm{i}ku_1(x,k) - u_1'(x,k)}{\mathrm{i}ku_1(x,k) + u_1'(x,k)},$$

and use (2.7) and (2.5).

Let $k \in \mathbb{R} \setminus \{0\}$ be fixed. We write $u_1(x,k)$ in terms of $u_2(x,k) = \mathrm{e}^{-\mathrm{i}kx} m_2(x,k)$, using (2.16) and get:

$$\mathrm{e}^{2\mathrm{i}kx} r(x,k) = \frac{2\mathrm{i}kR_2(k)m_2(x,k) - R_2(k)m_2'(x,k) - \overline{m}_2'(x,k)\mathrm{e}^{2\mathrm{i}kx}}{2\mathrm{i}k\overline{m}_2(x,k) + R_2(k)m_2'(x,k)\mathrm{e}^{-2\mathrm{i}kx} + \overline{m}_2'(x,k)}.$$

This gives (3.15), since $m_2(x,k) \to 1$ and $m_2'(x,k) \to 0$ when $x \to -\infty$ (see theorem 2.1). The equality (3.15) is obvious for $k = 0$, since $r(x,0) = R_2(0) = 0$ for every x.

The assertion in (iii) follows from the properties of w and from the fact that $\operatorname{Re} w > 0$. □

4. The transmission coefficient

We shall prove here that $R_2(k)$, $k \in \mathbb{R}$, uniquely determines the transmission coefficient $T(k)$, $k \in \mathbb{R}$. We have seen (section 2) that $T(\cdot)$ extends holomorphically to $\mathbb{C}^+$, is continuous on $\operatorname{Im} k = 0$ and $|T(k)| \leq 1$ when $k \in \mathbb{R}$.

We first obtain estimates for $T(k)$ when $k \in \overline{\mathbb{C}}^+$. The main result in this direction is contained in the next proposition.

Proposition 4.1. *Set $Q = 1 - \frac{1}{c}$. Then*

$$\left| T(k)\, \mathrm{e}^{\mathrm{i}k \int_{\mathbb{R}} Q} \right| \leq 1, \tag{4.1}$$

when $k \in \overline{\mathbb{C}}^+$.

To prove this proposition we need a lemma.

Lemma 4.2. *One has*

$$\big(2\kappa m_1(x,\mathrm{i}\kappa) - m_1'(x,i\kappa)\big)\mathrm{e}^{\kappa \int_x^\infty Q} \geq 2\kappa \tag{4.2}$$

when $x \in \mathbb{R}$ and $\kappa \geq 0$.

Proof. The estimate (4.2) is obvious for $\kappa = 0$, both sides being zero..

We fix $\kappa > 0$. Then since $w(x,\mathrm{i}\kappa)$ is real and

$$w(x,\mathrm{i}\kappa) = \frac{u_1' u_1}{-\kappa u_1^2}(x,\mathrm{i}\kappa) > 0,$$

we get $u_1' u_1(x,\mathrm{i}\kappa) < 0$. The function $u_1(x,\mathrm{i}\kappa)$ does not vanish at any point of $\mathbb{R}$ and is continuous, therefore it has constant sign on $\mathbb{R}$. Since $m_1(x,\mathrm{i}\kappa) = \mathrm{e}^{\kappa x} u_1(x,\mathrm{i}\kappa) \to 1$ when $x \to \infty$, $u_1(x,\mathrm{i}\kappa)$ must be

positive. Thus $u_1'(x,\mathrm{i}\kappa)<0$, and it follows that

$$m_1'(x,\mathrm{i}\kappa)<\kappa m_1(x,\mathrm{i}k).$$

We have that

$$u_1''(x,\mathrm{i}\kappa)-\frac{\kappa^2}{c^2(x)}u_1(x,\mathrm{i}\kappa)=0,$$

and therefore if we set $v(x)=u_1'(x,\mathrm{i}\kappa)u_1(x,\kappa)$, v satisfies

$$v'=\frac{\kappa^2}{c^2}u_1^2(\cdot,\mathrm{i}\kappa)+(u_1')^2(\cdot,\mathrm{i}\kappa)\geq -2\frac{\kappa}{c}v.$$

Hence $v\mathrm{e}^{2\kappa(x+\int_x^\infty Q)}$ is non-decreasing. We thus get that

$$v(x)\mathrm{e}^{2\kappa(x+\int_x^\infty Q)}\leq \lim_{x\to\infty}(u_1'u_1)(x,\mathrm{i}\kappa)\mathrm{e}^{2\kappa(x+\int_x^\infty Q)}=-\kappa,$$

that is,

$$-(u_1'u_1)(x,\mathrm{i}\kappa)\mathrm{e}^{2\kappa(x+\int_x^\infty Q)}\geq\kappa. \tag{4.3}$$

In (4.3) we replace $u_1(x,\mathrm{i}\kappa)$ by $m_1(x,\mathrm{i}\kappa)=e^{\kappa x}u_1(x,\mathrm{i}\kappa)$, and get

$$m_1(x,\mathrm{i}\kappa)[\kappa m_1(x,\mathrm{i}\kappa)-m_1'(x,\mathrm{i}\kappa)]\mathrm{e}^{2\kappa\int_x^\infty Q}\geq\kappa. \tag{4.4}$$

Since $m_1(x,\mathrm{i}\kappa)>0$, $\kappa m_1(x,\mathrm{i}\kappa)-m_1'(x,\mathrm{i}\kappa)>0$ and by (4.4), we obtain

$$\begin{aligned}\frac{2\kappa m_1-m_1'}{2\kappa}(x,\mathrm{i}\kappa)&=\frac{\kappa m_1-m_1'}{2\kappa}(x,\mathrm{i}\kappa)+\frac{m_1}{2}(x,\mathrm{i}\kappa)\\ &\geq\Big[\frac{m_1(\kappa m_1-m_1')}{\kappa}(x,\mathrm{i}\kappa)\Big]^{1/2}\geq\mathrm{e}^{-\kappa\int_x^\infty Q},\end{aligned}$$

and this completes the proof. □

Proof of Proposition 4.1. We show that $T(k)\,\mathrm{e}^{\mathrm{i}k\int_{\mathbb{R}}Q}$ has angular order less or equal to 1, and its absolute value is less then equal to 1 for $k\in\mathbb{R}$ (this is clear since $|T(k)|\leq 1$) and $k\in\mathrm{i}\mathbb{R}^+$. The inequality in the statement is then a consequence of the Phragmén-Lindelöf principle.

From (2.4) and (2.10) we have that

$$\begin{aligned}\frac{1}{T(k)}&=\frac{1}{2\mathrm{i}k}\Big[2\mathrm{i}k-k^2\int_{-\infty}^{\infty}q(t)m_1(t,k)\,\mathrm{d}t\Big]\\ &=\lim_{x\to-\infty}\frac{m_1'+2\mathrm{i}km_1}{2\mathrm{i}k}(x,k).\end{aligned}$$

We denote

$$f(x,k)=\frac{2\mathrm{i}k}{m_1'+2\mathrm{i}km_1}(x,k).$$

It follows that

$$T(k)\,\mathrm{e}^{\mathrm{i}k\int_{\mathbb{R}} Q} = \lim_{x\to-\infty} f(x,k)\mathrm{e}^{\mathrm{i}k\int_x^\infty Q}. \tag{4.5}$$

The equation (2.10) yields

$$\begin{aligned} f' &= -\frac{2\mathrm{i}k(m_1'' + 2\mathrm{i}km_1')}{(m_1' + 2\mathrm{i}km_1)^2} \\ &= \frac{\mathrm{i}kq}{2}\,\frac{2\mathrm{i}km_1}{(m_1' + 2\mathrm{i}k)}\,\frac{2\mathrm{i}k}{(m_1' + 2\mathrm{i}k)} \\ &= \frac{\mathrm{i}kq}{2}(r+1)f. \end{aligned} \tag{4.6}$$

The last equality has been consequence of the fact that $r+1 = \frac{2\mathrm{i}km_1}{(m'+2\mathrm{i}km_1)}$.

Since $\lim_{x\to\infty} f(x,k) = 1$, we deduce from (4.6) that

$$f(x,k) = \mathrm{e}^{-\mathrm{i}k\int_x^\infty \frac{q}{2}(r+1)}.$$

Thus

$$T(k)\,\mathrm{e}^{\mathrm{i}k\int_{\mathbb{R}} Q} = \mathrm{e}^{\frac{\mathrm{i}k}{2}\int_{\mathbb{R}}(Q^2(y) - q(y)r(y,k))\,\mathrm{d}y}, \tag{4.7}$$

where we have made use of the fact that $q = 2Q - Q^2$.

The equality (4.7) and the fact that $|r(x,k)| < 1$ ensures that

$$\mathbb{C}^+ \ni k \to T(k)\,\mathrm{e}^{\mathrm{i}k\int_{\mathbb{R}} Q} := h(k) \in \mathbb{C}$$

angular order less or equal to 1. On the other hand h if $\kappa \geq 0$, $h(\mathrm{i}\kappa)$ is positive (see (4.7), for instance), and (4.2) and (4.5) shows that $|h(\mathrm{i}\kappa)| \leq 1$ when $\kappa \geq 0$. This finishes the proof. □

The next theorem is the main result of this section.

Theorem 4.3. *For c satisfying [H1] and [H2], the transmission coefficient $(T(k))_{k\in\mathbb{R}}$ is uniquely determined by $(R_2(k))_{k\in\mathbb{R}}$.*

We shall need some lemmas.

Lemma 4.4. *We have*

$$\lim_{\kappa\to\infty} qr(\cdot,\mathrm{i}\kappa) = Q^2 \tag{4.8}$$

in $L^1(\mathbb{R})$.

Proof. (i) Assume first that $c \in C^\infty(\mathbb{R})$ and obeys [H1] and [H2]. We change coordinates

$$x(y) = \int_0^y \frac{1}{c(y)}\,\mathrm{d}s$$

and write $v(x(y),k) = u_1(y,k)$ (u_1 as in section 2). Then

$$r(y,k) = \Big(1 - \frac{1}{c(y)}\ \frac{v'(x(y),\mathrm{i}\kappa)}{\mathrm{i}\kappa\ v(x(y),\mathrm{i}\kappa)}\Big)\Big(1 + \frac{1}{c(y)}\ \frac{v'(x(y),\mathrm{i}\kappa)}{\mathrm{i}\kappa\ v(x(y),\mathrm{i}\kappa)}\Big)^{-1}.$$

Since $c \in C^\infty$, $v'(x(y),\mathrm{i}\kappa)/(\mathrm{i}\kappa\ v(x(y),\mathrm{i}\kappa))$ converges uniformly to 1 when $\kappa \to \infty$. (See [DT].) We get that, for y fixed,

$$\lim_{\kappa\to\infty} r(y,\mathrm{i}\kappa) = \frac{c(y)-1}{c(y)+1}.$$

It follows that $q(x)r(x,\mathrm{i}\kappa) \to Q^2(x)$ for every x, while (4.8) is a consequence of Lebesques's convergence theorem.

(ii) Assume now that c is a real measurable function that satisfies [H1] and [H2]. We set $c_\epsilon = 1 + (c-1)*\varphi_\epsilon$, şi $Q_\epsilon = 1 - c_\epsilon^{-2}$ where φ is a nonnegative smooth compactly supported function with $\int \varphi\ \mathrm{d}x = 1$, and $\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon})$. Then

$$c_\epsilon \geq 1 + (c_0 - 1)\int \varphi_\epsilon(y)\ \mathrm{d}y = c_0,$$

$$c_\epsilon \leq 1 + (c_M - 1)\int \varphi_\epsilon(y)\ \mathrm{d}y = c_M,$$

$$|q_\epsilon(x)| \leq \frac{2}{c_0}\int |c(y) - 1|\ \mathrm{d}y =: \gamma_1$$

and

$$\|q_\epsilon - q\|_{L^1} \leq \frac{1}{c_0^2}\|c_\epsilon - c\| \to 0 \quad \text{when } \epsilon \to 0.$$

We denote by w_ϵ and r_ϵ the functions defined as in (3.3) and (3.2) corresponding to c_ϵ. The, since $w_\epsilon(x,\mathrm{i}\kappa) > 0$, $w(x,\mathrm{i}\kappa) > 0$ and by lemma 3.5, it follows that there exists C independent of ϵ and κ (C may depend on c_0, c_M, γ_0 and γ_1) such that

$$\int |r_\epsilon(x,\mathrm{i}\kappa) - r(x,\mathrm{i}\kappa)|\ \mathrm{d}x \leq \int |w_\epsilon(x,\mathrm{i}\kappa) - w(x,\mathrm{i}\kappa)|\ \mathrm{d}x \leq C\|q_\epsilon - q\|_{L^1}.$$

We obtain

$$\begin{aligned}\int |q(x)r(x,\mathrm{i}\kappa) - Q^2(x)|\ \mathrm{d}x &\leq \int |q(x)r(x,\mathrm{i}\kappa) - q_\epsilon(x)r_\epsilon(x,\mathrm{i}\kappa)|\ \mathrm{d}x \\ &+ \int |Q_\epsilon^2(x) - Q^2(x)|\ \mathrm{d}x + \int |Q_\epsilon^2(x) - q_\epsilon(x)r_\epsilon(x,\mathrm{i}\kappa)|\ \mathrm{d}x \\ &\leq C\|c - c_\epsilon\|_{L^1} + \int |Q_\epsilon^2 - q_\epsilon(x)r_\epsilon(x,\mathrm{i}\kappa)|\ \mathrm{d}x,\end{aligned}$$

C constant independent of κ. Let $\delta > 0$ be fixed. There exists $\epsilon_0 = \epsilon_0(\delta)$ with $C\|c - c_\epsilon\|_{L^1} \leq \delta/2$. On the other hand, the discussion in (i) shows that there exists $\kappa_0 = \kappa_0(\delta)$ such that if $\kappa \geq \kappa_0$ then

$$\int |Q_\epsilon^2 - q_\epsilon(x)r_\epsilon(x,\mathrm{i}\kappa)|\ \mathrm{d}x \leq \delta/2.$$

Hence, if $\kappa \geq \kappa_0 = \kappa_0(\delta)$, we have

$$\int |q(x)r(x,\mathrm{i}\kappa) - Q^2(x)|\ \mathrm{d}x \leq \delta\ .$$

Since δ has been arbitrarily chosen, this completes the proof of the lemma. $\square$

Lemma 4.5. *The quantity*

$$\int_{\mathbb{R}} Q(x)\ \mathrm{d}x$$

is uniquely determined by the reflexion coefficient.

Proof. Consider $k \in \mathbb{R} \setminus \{0\}$ be fixed. We multiply the equation

$$r' = -2\mathrm{i}kr + \frac{\mathrm{i}kq}{2}(1+r)^2,$$

by $\overline{r}$, and the take real parts. We obtain

$$(1-|r|^2)' = \frac{ikq}{2}(1-|r|^2)(r-\overline{r}),$$

and since $\lim_{x\to\infty} |r(x,k)|^2 = 0$ this yields

$$1 - |r(x,k)|^2 = \mathrm{e}^{-\int_x^\infty \frac{\mathrm{i}kq(y)}{2}(r-\overline{r})(y,k)\ \mathrm{d}y}. \tag{4.9}$$

we let x to $-\infty$ in (4.9) and use (3.15). It follows that

$$1 - |R_2(k)|^2 = \mathrm{e}^{-\int_{\mathbb{R}} \frac{\mathrm{i}kq(y)}{2}(r-\overline{r})(y,k)\ \mathrm{d}y} \qquad k \in \mathbb{R}. \tag{4.10}$$

Hence $k^2 \int_{\mathbb{R}} q\mathrm{Im}\,(r/k)\,\mathrm{d}y$ is uniquely determined by $R_2(k)$ for $k \in \mathbb{R}$, and therefore $\int_{\mathbb{R}} q\mathrm{Im}\,(r/k)\,\mathrm{d}y$ is also uniquely determined by $R_2(k)$.

Consider the harmonic function

$$h(k) = \int_{\mathbb{R}} q(y)\mathrm{Im}\,\big(\frac{r(y,k)}{k}\big)\ \mathrm{d}y.$$

It is bounded and continuous on $\mathrm{Im}\,k \geq 0$. Hence (see [Koo]) h is uniquely determined by its values at $\{\mathrm{Im}\,k = 0\}$, therefore by $R_2(k)$. Because $r(x, \mathrm{i}\kappa)$ is real when $\kappa \geq 0$, we have

$$h(\mathrm{i}\kappa) = -\frac{1}{\kappa}\int_{\mathbb{R}} q(x)r(x,\mathrm{i}\kappa)\ \mathrm{d}x.$$

It follows that $\int q(x)r(x,\mathrm{i}\kappa)\ \mathrm{d}x$ is uniquely determined by $R_2(k)$. Lemma 4.4 gives now that $\int Q^2(x)\ \mathrm{d}x$ is uniquely determined by $R_2(k)$.

The lemma is now a consequence of the equality $2Q = Q^2 + q$, and of the fact that $\int q(x)\ \mathrm{d}x$ is uniquely given by the limit of $2\mathrm{i}R(k)/k$ when $k \to 0$ (see lemma 2.2). □

Proof of theorem 4.3. When $k \in \overline{\mathbb{C}}^+$ we denote

$$F(k) = T(k)\mathrm{e}^{\mathrm{i}k\int Q\ \mathrm{d}x}. \tag{4.11}$$

We notice that $|F(k)|^2 = |T(k)|^2 = 1 - |R_2(k)|^2$ when k real, hence $|F(k)|$, $k \in \mathbb{R}$ is uniquely determined by $R_2(k)$. In addition F belongs to $H^\infty(\mathbb{C}^+)$ (proposition 4.1), is continuous on $\{\mathrm{Im}\,k = 0\}$ and has no

zeros. Moreover, it follows from (4.7) that

$$F(k)=\exp\Big(-\mathrm{i}k\int\limits_{\mathbb{R}} q(y)\frac{r}{2}(y,k)\ \mathrm{d}y+\mathrm{i}k\int\limits_{\mathbb{R}}\frac{Q(y)^2}{2}\ \mathrm{d}y\Big),$$

hence from lemma 4.4 we see that

$$\lim_{\kappa\to\infty}\frac{\log|F(\mathrm{i}\kappa)|}{\kappa}=0.$$

Then the factorization theorem of $H^\infty(\mathbb{C})$ functions (see [Koo]) gives that

$$F(z)=\gamma\Theta_F(z),\quad z\in\mathbb{C}^+, \tag{4.12}$$

where $\gamma\in\mathbb{C}$, $|\gamma|=1$, and Θ_F is the exterior function corresponding to F,

$$\Theta_F(z)=\exp\big(-\frac{\mathrm{i}}{2\pi}\int\limits_{\mathbb{R}}\big(\frac{1}{k-z}-\frac{k}{k^2+1}\big)\log|F(k)|^2\,\mathrm{d}k\big).$$

Since $|F(k)|^2=|T(k)|^2=1-|R(k)|^2$ when $k\in\mathbb{R}$, we may write

$$\Theta_F(z)=\exp\big(-\frac{\mathrm{i}}{2\pi}\int\limits_{\mathbb{R}}\big(\frac{1}{k-z}-\frac{k}{k^2+1}\big)\log|F(k)|^2\,\mathrm{d}k\big). \tag{4.13}$$

When k is real $|F(k)|=|T(k)|$, hence form (4.6) it follows that $\log|F(k)|^2/k$ is locally L^1. We write

$$\int\limits_{\mathbb{R}}\big(\frac{1}{k-z}-\frac{k}{k^2+1}\big)\log|F(k)|^2\,\mathrm{d}k=$$
$$=\int\limits_{\mathbb{R}}\frac{zk+1}{(k-z)(k^2+1)}\ \log|F(k)|^2\,\mathrm{d}k$$

and compute

$$\lim_{t\to0}\int\limits_{\mathbb{R}}\big(\frac{1}{k-\mathrm{i}t}-\frac{k}{k^2+1}\big)\log|F(k)|^2\,\mathrm{d}k=\int\limits_{\mathbb{R}}\frac{1}{k(k^2+1)}\log|F(k)|^2\,\mathrm{d}k=0,$$

where in the last equality we have used that the integrand is an odd function. (Recall that $\overline{T(k)}=T(-k)$ when k is real.) We thus obtain that $\Theta_F(0)=1$.

Writing (4.12) at $z=0$ we easily get that $\gamma=1$, since $F(0)=1$ and $\Theta_F(0)=1$.

It follows from (4.11), (4.12), (4.13) and the previous lemma that $T(k)$ is uniquely determined by $R_2(k)$. □

5. Further properties of $m_1(x,k)$.

We recall that a holomorphic function f on $\mathbb{C}^+$ is said to belong to the Smirnov class $\mathcal{N}_+$ if it can be represented in the form $f=h^{-1}g$, where h, $g\in H^\infty(\mathbb{C}^+)$ and h is exterior.

If $f\in\mathcal{N}_+$ then the boundary values

$$f(\lambda)\lim_{\varepsilon\to0}f(\lambda+\mathrm{i}\epsilon)$$

exist almost everywhere $\lambda \in \mathbb{R}$, and f can be recovered by these boundary values, therefore one can identify f with its boundary values at $\operatorname{Im} z = 0$. From the Smirnov maximum principle we aslo have that

$$L^2(\mathbb{R}) \cap \mathcal{N}_+ = H^2(\mathbb{C}^+).$$

(See [Koo].)

We use the notation

$$\tilde{T}(k) = T(k)e^{\mathrm{i}k\int Q}. \tag{5.1}$$

We recall that $\tilde{T}$ is an $H^\infty(\mathbb{C}^+)$ exterior function (see the proof of theorem 4.3).

Lemma 5.1. *Assume $x \in \mathbb{R}$ is fixed.*

a) *The function*

$$\mathbb{C}^+ \ni k \to m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q} \in \mathbb{C}$$

belongs to the Smirnov class.

b) *The function*

$$\mathbb{C}^+ \ni k \to \frac{\tilde{T}(k)(m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q} - 1)}{k} \in \mathbb{C}$$

belongs to the Hardy space $H^2(\mathbb{C}^+)$.

Proof. We first show that g given by

$$g(k) = m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q}$$

belongs to the Smirnov class $\mathcal{N}_+$.

For that, we notice that if $T_x(k)$ is the transmission coefficient associated to

$$c_x(y) = \begin{cases} c(y), & y \geq x \\ 1, & y < x \end{cases}$$

then (from (2.18))

$$\frac{2\mathrm{i}k}{T_x(k)} = u_1'(x,k)\mathrm{e}^{-\mathrm{i}kx} + \mathrm{i}k\mathrm{e}^{-\mathrm{i}kx}u_1(x,k), \qquad k \in \overline{\mathbb{C}}^+,$$

hence

$$2(T_x(k)u_1(x,k)e^{-\mathrm{i}kx})^{-1} = w(x,k) + 1.$$

It follows that

$$T_x(k)m_1(x,k) = r(x,k) + 1 \quad \text{when } k \in \overline{C}^+,$$

and thus

$$m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q} = (1 + r(x,k))(T_x(k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q})^{-1}$$

belongs to $\mathcal{N}_+$ since $1 + r(x, \cdot)$ is an $H^\infty(\mathbb{C}^+)$ function, while $T_x(k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q}$ is an $H^\infty(\mathbb{C}^+)$ exterior function.

Since $T(k)\mathrm{e}^{-\mathrm{i}k\int Q}$ is $H^\infty(\mathbb{C}^+)$ and exterior we get that $f(k) = \tilde{T}(k)m_1(x,k)\mathrm{e}^{-\mathrm{i}\kappa\int_x^\infty Q}$ belongs $\mathcal{N}_+$.

By using Smirnov maximum principle and (2.5) we see that it suffices to show that $f/(\mathrm{i}+k)$ is L^2 on $\mathbb{R}$. We notice that lemma 2.3 leads to

$$|f(k)/(\mathrm{i}+k)|^2 \leq 2\mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k))/(1+k^2).$$

On the other hand it follows form (2.18) that

$$\mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k)) = \mathrm{Re}\,\Big(\frac{1}{w(x,k)+w_-(x,k)}\Big),$$

and this along with corollary 3.3 gives that $k \to \mathrm{Re}\,(T(k)m_1(x,k)m_2(x,k))/(1+k^2)$ is L^1. We have obtained that $f/(\mathrm{i}+k)$ is $L^2(\mathbb{R})$, which completes the proof. □

6. The uniqueness

We prove here theorem 1.1.

Let $c\colon \mathbb{R} \to \mathbb{R}$ a real measurable function obeying [H1] and [H2] and satisfying also the additional assumption

$$c' \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \tag{6.1}$$

We need some results that are valid in this more particular case.

We consider the coordinate transform

$$y = \chi(x) = x + \int_x^\infty Q(y)\,\mathrm{d}y, \quad Q = 1 - 1/c,$$

($\chi'(x) = 1/c(x)$ is positive and continuous.) If u is a solution to

$$u'' + \frac{k^2}{c^2}u = 0, \tag{6.2}$$

we set

$$v(y) = u(\chi^{-1}(y)). \tag{6.3}$$

Then

$$c(x)u'(x) = v'(\chi(x)), \tag{6.4}$$

and if $\mu(x) = 1/c(\chi^{-1}(y))$, then v solves the equation

$$\frac{1}{\mu}(\mu v')' + k^2 v = 0.$$

It is easily seen that μ satisfies [H1], [H2] and $\mu' \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

If $u_1(x,k)$, $k \in \overline{\mathbb{C}}^+$ is the Jost solution constructed in section 2, then

$$v_1(y,k) = u_1(\chi^{-1}(y),k).$$

satisfies $v_1(x,k) - \mathrm{e}^{\mathrm{i}kx} \to 0$ when $x \to \infty$. We then may define

$$\rho(y,k) = \frac{1 - v_1'(y,k)/\mathrm{i}kv_1(y,k)}{1 + v_1'(y,k)/\mathrm{i}kv_1(y,k)}. \tag{6.5}$$

(See [Br00].)

The following important result is due to J. Sylvester (see, for instance, Theorem 6.2 [Br00]).

Theorem 6.1. *Assume that c is a real measurable function satisfying [H1], [H2] and* (6.1). *Then for the function ρ defined above one has that*

$$|\rho(x,k)| \le 1 - \mathrm{e}^{-\int_{\mathbb{R}} |\mu'/\mu|\,\mathrm{d}s}$$

when x, $k \in \mathbb{R}$.

We write (6.5) at the point $\chi(x)$ and recall that (6.4) holds. We thus obtain

$$\rho(\chi(x),k) = \frac{\mathrm{i}ku_1(x,k) - c(x)u_1'(x,k)}{\mathrm{i}ku_1(x,k) + c(x)u_1'(x,k)}.$$

It follows from theorem 6.1 and (2.13) that there exists a constant $a < 1$ such that

$$k^2|u_1|^2 + c^2|u_1'| - 2k^2c \le a(k^2|u_1|^2 + c^2|u_1'| + 2k^2c). \tag{6.6}$$

We get the following lemma.

Lemma 6.2. *Assume c satisfies [H1], [H2] and* (6.1). *Then there exists a constant $M > 0$ with*

$$\left|m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q}\right| \le M \tag{6.7}$$

when $x \in \mathbb{R}$ and $k \in \overline{\mathbb{C}}^+$.

Proof. When $k \in \mathbb{R}$,

$$|m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q}| = |m_1(x,k)| = |u_1(x,k)|$$

for every x, and (6.7) follows (6.6). The lemma follows from 5.1 and the Smirnov maximum principle. □

Corollary 6.3. *Assume c satisfies [H1], [H2] and* (6.1). *Then the function*

$$\mathbb{C}^+ \ni k \to \frac{m_1(x,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty Q} - 1}{k} \in \mathbb{C}$$

belongs to the Hardy space $H^2(\mathbb{C}^+)$.

Proof. The statement follows from the previous lemma and (2.5). □

Proof of theorem 1.1. Let c_1 and c_2 be two functions that satisfy [H1], [H2] and (6.1) and such that

$$R_2(k;c_1) = R_2(k;c_2) = R_2(k) \quad k \in \mathbb{R}.$$

Then

$$T(k;c_1) = T(k,c_2) = T(k) \text{ and } \int_{\mathbb{R}} Q_1 \,\mathrm{d}s = \int_{\mathbb{R}} Q_2 \,\mathrm{d}s.$$

(See theorem 4.3 and lemma 4.5.)

We denote by $u_{1,1}(x,k)$, $u_{2,1}(x,k)$ the Jost solutions corresponding to c_1 and by $u_{2,1}(x,k)$, $u_{2,2}(x,k)$ those corresponding to c_2. We write $u_{1,j}(x,k) = \mathrm{e}^{\mathrm{i}kx} m_{1,j}(x,k)$, $u_{2,j}(x,k) = \mathrm{e}^{-\mathrm{i}kx} m_{2,j}(x,k)$, $j = 1,2$. With this notation (2.17) becomes

$$T(k) m_{1,j}(x,k) = R_2(k)\mathrm{e}^{-2\mathrm{i}kx} m_{2,j}(x,k) + \overline{m_{2,j}(x,k)}, \; j = 1,2 \tag{6.8}$$

when k is real. We set

$$\tilde{T}(k) = \mathrm{e}^{\mathrm{i}k\int Q_1} T(k) = \mathrm{e}^{\mathrm{i}k\int Q_2} T(k), \; \tilde{m}_{1,j} = \mathrm{e}^{-\mathrm{i}k\int_x^\infty Q_j} m_{1,j}, \; \tilde{m}_{2,j} = \mathrm{e}^{-\mathrm{i}k\int_{-\infty}^x Q_j} m_{2,j}.$$

We multiply (6.8) by $\mathrm{e}^{\mathrm{i}k\int Q}$ and get

$$\tilde{T}(k)\tilde{m}_{1,j}(x,k) = R_2(k)\mathrm{e}^{-2\mathrm{i}k(x-\int_{-\infty}^x Q_j)} \tilde{m}_{2,j}(x,k) + \overline{\tilde{m}_{2,j}(x,k)}, \; j = 1,2. \tag{6.9}$$

We change coordinates

$$y = \chi_j(x) = x - \int_{-\infty}^{x} Q_j(s)\,\mathrm{d}s$$

and denote

$$M_{j,l}(y) = \tilde{m}_{j,l}(\chi_l^{-1}(y)), \quad j,l = 1,2.$$

Then (6.9) can be written as

$$\tilde{T}(k) M_{1,j}(y,k) = R_2(k)\mathrm{e}^{-2\mathrm{i}ky} M_{2,j}(y,k) + \overline{M_{2,j}(y,k)}, \quad j = 1,2. \tag{6.10}$$

If

$$M_1(y,k) = \frac{1}{k}(M_{1,1}(y,k) - M_{1,2}(y,k)), \quad M_2(y,k) = \frac{1}{k}(M_{2,1}(y,k) - M_{2,2}(y,k)),$$

it follows that

$$\tilde{T}(k) M_1(y,k) = R_2(k)\mathrm{e}^{-2\mathrm{i}kx} M_2(y,k) + \overline{M_2(y,k)}. \tag{6.11}$$

We multiply this relation by $M_2(y,k)$ and get

$$\tilde{T}(k) M_1(y,k) M_2(y,k) = R_2(k)\mathrm{e}^{-2\mathrm{i}kx} M_2^2(y,k) + |M_2(y,k)|^2. \tag{6.12}$$

It follows from corollary 6.3 and from lemma 5.1 that $\tilde{T}(\cdot)M_1(y,\cdot)M_2(y,\cdot)$ belongs to $H^1(\mathbb{C}^+)$, therefore

$$\int_{\mathbb{R}} \tilde{T}(k) M_1(y,k) M_2(x,k)\,\mathrm{d}k = 0. \tag{6.13}$$

On the other hand, we easily get from (6.10) (proceeding as in the proof of lemma 2.3) that

$$\operatorname{Re}\tilde{T}(k)M_1(y,k)M_2(x,k)) = \frac{1}{2}\big(|\tilde{T}(k)M_1(y,k)|^2 + |\tilde{T}(k)M_2(y,k)|^2\big), \tag{6.14}$$

when k is real. Hence (6.13), (6.14) and the fact that $M_1(y,\cdot)$ $M_2(y,\cdot)$ are holomorphic yield

$$M_1(y,k) = 0, \quad M_2(y,k) = 0 \quad \text{when } y \in \mathbb{R} \text{ and } k \in \overline{\mathbb{C}}^+.$$

Therefore

$$m_{j,1}(\chi_1^{-1}(y),k) = m_{j,2}(\chi_2^{-1}(y),k) \quad \text{when } y \in \mathbb{R} \text{ and } k \in \overline{\mathbb{C}}^+, \text{ j=1, 2.}$$

We let k go to zero in $(m_{2,j}(\chi_j^{-1}(y),k)-1)/k$. We get that

$$\int_{-\infty}^{\chi_1^{-1}(y)} Q_1(s)\,\mathrm{d}s = \int_{-\infty}^{\chi_2^{-1}(y)} Q_2(s)\,\mathrm{d}s$$

for every y. We change variable $s = \chi_1^{-1}(t)$ in the integral in left hand side and $s = \chi_2^{-1}(t)$ in the integral in the right hand side and then take derivative with respect to y. We obtain that

$$c_1(\chi_1^{-1}(y)) = c_2(\chi_2^1(y))$$

for every y, that is $\chi_1'(y) = \chi_2'(y)$ almost everywhere, and this equality is nothing else than

$$\frac{1}{c_1(y)} = \frac{1}{c_2(y)},$$

and the proof of the theorem is complete. $\square$

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